INDE 597
Homework 4
Due date 3/27

In general, solving an IP is NP-Hard, while LPs can be solved in polynomial time.

Given an IP $\max\{c^T x: Ax \leq b, x \in \mathbb{Z}_+^n\}$, if we are lucky, the optimal solution of the LP relaxation $\max\{c^T x: Ax \leq b, x \in \mathbb{R}_+^n\}$ will be integral, and hence that solution will be equal to the optimal solution of $\max\{c^T x: Ax \leq b, x \in \mathbb{Z}_+^n\}$. This allows us to solve an IP at the cost of solving an LP.

It would be nice to have a sufficient condition on the constraint matrix A that guarantees that the LP relaxation will have an integral optimal solution. A matrix A is *totally unimodular* (*TU*) if every square submatrix of A has determinant 0, 1, or -1.

1) Show that $\max\{c^T x: Ax \leq b, x \in \mathbb{R}_+^n\}$ (where all entries of A are integers) has an integral optimal solution for all integer vectors b for which it has a finite optimal value if and only if A is TU.

Thus, IPs whose constraint matrices are TU can be solved in polynomial time, by solving the LP relaxation. We will now look at some properties and special cases of TU matrices.

2) Show that:
   a) A is TU iff A^T is TU iff $[A\ I]$ is TU.
   b) The (signed) incidence matrix of a directed graph is TU.
   c) The incidence matrix of a bipartite graph is TU.

3) Given a network $G = (V, E)$ with source x, sink y, and integral capacities $c_{uv}\ \ \forall (u, v) \in E$, the max flow between x and y can be found as follows:

   Add an edge (y, x) to G with $c_{yx} = \infty$, let f_{uv} be the flow along edge (u, v), and compute
   $\max\{f_{yx} : \sum_{u \in \partial^+(v)} f_{vu} = \sum_{u \in \partial^-(v)} f_{uv}$ for $v \in V, 0 \leq f_{uv} \leq c_{uv}$ for $(u, v) \in E\}$.

   a) Show that the dual of this LP models the problem of finding a minimum cut between x and y.
   b) Show that the max-flow and min-cut problems are strongly dual.
   c) Show that the constraint matrix is TU and conclude that both problems can be solved in polynomial time.
4) Let X be a set of workers and Y be a set of jobs, where each worker is trained to complete some subset of the jobs. Consider the problem of assigning jobs to workers so that each worker does a job they are trained for, and as many workers as possible have jobs.
   a) Give a combinatorial algorithm for this problem, using the Ford-Fulkerson algorithm as a subroutine.
   b) Formulate this problem as an IP and show that the IP can be solved in polynomial time.